CARTAN CONNECTION AND CURVATURE FORMS

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1 Connections and Riemannian Curvature

Cartan formula is a way of generalising the Riemannian curvature for a Riemannian manifold (M, g) of dimension n to a differentiable manifold. We start by defining the Riemannian manifold and curvatures. We begin by defining the usual notion of connection and the Levi-Civita connection.

1.1 Affine Connection

Definition 1.1 (Affine connection)**.** Let M be a differentiable manifold and E a vector bundle over M. A connection or covariant derivative at a point $p \in M$ is a map $D : \Gamma(E) \to \Gamma(T^*M \otimes E)$ with the properties for any $V, W \in T_pM, \sigma, \tau \in \Gamma(E)$ and $f \in C^\infty(M)$, we have that $D_V\sigma \in E_p$ with the following properties:

1. D is tensorial over T_pM i.e.
$$D_{fV+W}\sigma = fD_V\sigma + D_W\sigma.$$
2. D is $\mathbb{R}$-linear in $\Gamma(E)$ i.e.
$$D_V(\sigma + \tau) = D_V\sigma + D_V\tau.$$
3. D satisfies the Leibniz rule over $\Gamma(E)$ i.e. if $f \in C^\infty(M)$, we have
$$D_V(f\sigma) = df(V) \cdot \sigma + fD_V\sigma.$$

Note that we did not require any condition of the metric g on the manifold, nor the bundle metric on E in the abstract definition of a connection.

Remark 1.1. Given connections D^E and D^F on vector bundles E and F respectively, we can define a connection $D^{E\otimes F}$ on the tensor bundle of $E \otimes F$ by:

$$D_X^{E\otimes F}(\sigma \otimes \mu) = (D_X^E \sigma) \otimes \mu + \sigma \otimes (D_X^F \mu).$$

Similarly, a connection D^E on vector bundle E induces a connection D^{E^*} on E^* by using the product rule and connection on E. Explicitly, if $\sigma \in E$ and $s \in E^*$, we have the following:

$$X(s(\sigma)) = (D_X^{E^*} s)(\sigma) + s(D_X^E \sigma).$$

Concretely, any affine connection is fully defined by the values of $D\sigma \in \Gamma(T^*M \otimes E)$ for all $\sigma \in E$. We can choose a basis $\{\sigma_i\}_{i=1}^n$ of E and define the values of $D\sigma_i$ for all $i = 1, \ldots, n$. Therefore, for a given vector bundle on M, there are infinitely many choices of connections on any given vector bundle E. And one of the special one in the study of Riemannian geometry is the Levi-Civita connection.

1.2 Levi-Civita Connection and Riemannian Curvature

Next, we can endow the vector bundle E with a bundle metric which is a family of inner products on the fibres of E_x for all $x \in M$ which varies smoothly on $x \in M$. In tensorial notation, a metric is an element of the symmetric 2-tensor $\langle \cdot, \cdot \rangle \in \Gamma(\mathrm{Sym}^2(E^*))$ which is positive definite. If E is the tangent bundle TM, then this bundle metric is just the Riemannian metric g.

Definition 1.2 (Compatible connection)**.** A connection D on a vector bundle E is compatible with the metric $\langle \cdot, \cdot \rangle$ on E if for any $X \in \Gamma(TM)$ and $\sigma, \mu \in \Gamma(E)$, we have:

$$D_X \langle \sigma, \mu \rangle = X\langle \sigma, \mu \rangle = \langle D_X \sigma, \mu \rangle + \langle \sigma, D_X \mu \rangle$$

Theorem 1.1 (Fundamental theorem of Riemannian geometry)**.** On each Riemannian manifold (M, g), there exists a unique connection ∇ on TM which is compatible with the metric (i.e. $\nabla g \equiv 0$) and torsion-free (i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$ for any vector fields X and Y). This connection is called the Levi-Civita connection.

For vector fields $X, Y \in \Gamma(TM)$, in local coordinates $\{x^i\}$, we can write them explicitly as:

$$X = \sum_{i=1}^m X^i(x)\partial_{x_i} \quad \text{and} \quad Y = \sum_{i=1}^m Y^i(x)\partial_{x_i},$$

where $X^i(x)$ and $Y^i(x)$ are smooth functions on M for all $i = 1, 2, \ldots, m$. Thus, the expression for $\nabla_X Y$ can be computed directly by the properties of a connection in Definition 1.1 as follows:

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^i \partial_{x_i}}(Y^j \partial_{x_j}) \\
&= X^i \nabla_{\partial_{x_i}}(Y^j \partial_{x_j}) \\
&= (X^i \partial_{x_i} Y^j)\partial_{x_j} + X^i Y^j (\nabla_{\partial_{x_i}} \partial_{x_j}).
\end{aligned}$$

In the above expression, all of the terms are defined, except for $\nabla_{\partial_{x_i}} \partial_{x_j} \in \Gamma(TM)$. This can actually be written in terms of a collection of objects called the Christoffel symbols Γ_{ij}^k i.e.

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \Gamma_{ij}^k \partial_{x_k} \quad \text{for all} \quad i, j, k \in \{1, 2, \ldots, n\}. \tag{1}$$

Therefore, a Levi-Civita connection is fully characterised by these symbols. In local coordinates, the Christoffel symbols can be computed explicitly in terms of the metric g using the torsion free and compatibility properties. It is given as follows:

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} \left(\partial_{x_j} g_{mi} + \partial_{x_i} g_{mj} - \partial_{x_m} g_{ij} \right). \tag{2}$$

Note the symmetry between the indices i and j i.e. $\Gamma_{ij}^k = \Gamma_{ji}^k$. Therefore, the Christoffel symbols is simply a collection of $\frac{m^2(m+1)}{2}$ functions (which are fixed by the metric g) on the manifold (M, g). And thus, the Levi-Civita connection is completely defined by this choice of functions.

With this out of the way, we can define the second covariant derivative, which in turn will be used to define curvature of a metric with this unique connection. The second covariant derivative is simply the composition of two covariant derivatives on a section of a vector bundle. This is defined via the product rule.

Definition 1.3 (Second covariant derivative)**.** We define the second covariant derivative on a vector bundle E as the map $D^2 : \Gamma(E) \to \Gamma(T^*M \otimes T^*M \otimes E)$ given by

$$D^2_{XY}\sigma = (D^2\sigma)(X, Y) = (D_X D\sigma)(Y) = D_X(D_Y\sigma) - D_{\nabla_X Y}\sigma$$

where ∇ is the Levi-Civita connection. This expression is $C^\infty(M)$-linear over X and Y.

Unlike the second order derivative in Euclidean space, the second covariant derivatives generally do not commute. In other words, we get the following:

$$D^2\sigma(X, Y) - D^2\sigma(Y, X) = D_X(D_Y\sigma) - D_Y(D_X\sigma) - D_{[X,Y]}\sigma.$$

The expression above measures the failure for the second covariant derivative to commute. We can easily show that this quantity is in fact a tensor. This quantity is called the curvature of the connection, which is a central object of study in Riemannian geometry.

1.3 Riemannian Curvature

Definition 1.4 (Curvature)**.** The curvature of the connection D on the vector bundle E is the tensor $R \in \Gamma(T^*M \otimes T^*M \otimes E^* \otimes E)$ given by, for any $X, Y \in \Gamma(TM)$ and $\sigma \in \Gamma(E)$:

$$\begin{aligned} R(X, Y)\sigma &= D^2\sigma(X, Y) - D^2\sigma(Y, X) \\ &= D_X(D_Y\sigma) - D_Y(D_X\sigma) - D_{[X,Y]}\sigma \in \Gamma(E). \end{aligned} \tag{3}$$

Definition 1.5 (Riemann curvature)**.** When D is the Levi-Civita connection ∇ of Riemannian metric g (and hence E is TM), the quantity R in Definition 1.4 is called the Riemann curvature. In local coordinates $\{x^i\}_{i=1}^n$, we can write the Riemann curvature as:

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}.$$

In fact, the quantity $R_{ijk}{}^l$ can be calculated explicitly since we know all the Christoffel symbols:

$$R_{ijk}{}^l = \partial_{x_j}\Gamma_{ik}^l - \partial_{x_i}\Gamma_{jk}^l + \Gamma_{jm}^l\Gamma_{ik}^m - \Gamma_{im}^l\Gamma_{jk}^m.$$

We also have the Riemann curvature tensor, also denoted R, which is defined as:

$$R(X, Y, Z, W) = g(R(X, Y)Z, W),$$

for all $W, X, Y, Z \in \Gamma(TM)$. In coordinates, the tensor can be written as:

$$R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

where $R_{ijkl} = g_{lm} R_{ijk}{}^m$.

2 Covariant External Derivative

Of course, in the previous section, the Levi-Civita connection and the Riemannian curvature can be written explicitly in local coordinates. However, we want to find a more general form of connections and curvature without the presence of a metric by using covariant external derivative.

The Riemannian curvature depends on the Levi-Civita connection for $\nabla_X Y$ to make sense. In particular, we need to know how to differentiate a tangent bundle by setting a connection on T^*M using the definitions/formulas (1) and (2). However, without a metric, we need an analogue of how to define this quantity for a general differentiable manifold.

Recall that a connection must satisfy the Leibniz rule:

$$D(f\sigma) = df\sigma + fD\sigma,$$

for any smooth function f on M and $\sigma \in \Gamma(E)$. Note that the d is just the exterior derivative on a 0-form. We can extend this definition onto higher degree differential forms $\Omega^k(M)$ for $k = 1, \ldots, n$. In particular, we are interested for the case of $k = 1$ which enables us to define the idea of curvature with respect to a connection. But first, let us define the connection.

2.1 Connection Form

Definition 2.1 (Covariant external derivative)**.** Let E be a tensor bundle on M. A covariant external derivative (or external connection) is a map sending E-valued k-forms to E-valued $k+1$-forms by the following rule:

$$\begin{aligned} D : \Omega^k(M, E) = \Gamma(\Lambda^k(T^*M) \otimes E) &\to \Omega^{k+1}(M, E) \\ w \otimes e &\mapsto dw \otimes e + (-1)^k w \wedge De, \end{aligned}$$

where $De \in \Omega(M, E) \cong \Gamma(T^*M \otimes E)$.

Of course, as we have seen in the previous section, all of the terms in a covariant derivative is defined except for the terms of the form $De \in \Gamma(T^*M \otimes E)$. Given an external connection D on the vector bundle E as defined in Definition 2.1 i.e. $D : \Gamma(E) \to \Gamma(T^*M \otimes E)$, we want to find an expression of this connection by defining how D acts on elements $e \in E$, or sufficiently, on the basis $\{e_i\}_{i=1}^n$ of E. We define the connection form as follows:

Definition 2.2 (Connection forms)**.** A connection form w_i^j on M is an $n \times n$ matrix of one forms T^*M such that for a basis $\{e_i\}$ for E, we have

$$De_i = \sum_{j=1}^n w_i^j \otimes e_j \in \Gamma(T^*M \otimes E). \tag{4}$$

Thus, for any $X \in \Gamma(TM)$, we have the following expression for a connection:

$$D_X e_i = \sum_{j=1}^n w_i^j(X) e_j \in \Gamma(E).$$

The choice for the matrix of one forms ω_i^j fully characterises the connection on E. Therefore, again, we have infinitely many different external connections on a given vector bundle E on M.

Remark 2.1. One property of the connection forms is that it is anti-symmetric i.e. $w_i^j + w_j^i = 0$.

By properties of connections in Definition 1.1, we can extend this using Leibniz rule and linearity so that for $\alpha \in \Gamma(E)$, we can write $\alpha = \sum_{i=1}^{n} \alpha^i e_i$ and thus:

$$D\alpha = \sum_{i=1}^{n} D(\alpha^i e_i) = \sum_{k=1}^{n} d\alpha^k \otimes e_k + \sum_{j,k=1}^{n} \alpha^j w_j^k \otimes e_k = d\alpha + w\alpha. \tag{5}$$

2.2 Curvature Form

We can next define the curvature on a vector bundle E as the second covariant exterior derivative of (5). Note that the exterior covariant derivative on 1-forms is just the external derivative. From above, we have that for any $\alpha \in \Gamma(E)$, we have $D\alpha = d\alpha + w\alpha$. By replacing E with $T^*M \otimes E$ in the above and taking the second exterior derivative of this, we get

$$\begin{aligned} D(D\alpha) &= D(d\alpha + w\alpha) = d(d\alpha + w\alpha) + w(d\alpha + w\alpha) \\ &= \cancel{d^2\alpha} + dw \otimes \alpha - w \wedge d\alpha + w \wedge d\alpha + (w \wedge w)\alpha = (dw + w \wedge w)\alpha. \end{aligned}$$

This quantity is called the curvature form, denoted Ω. Formally, we have the following definition:

Definition 2.3 (Curvature form)**.** A curvature form Ω is an E valued 2-form defined as $\Omega = D \circ D : \Gamma(E) \to \Gamma(\bigwedge^2 T^*M \otimes E)$ is a matrix of 2-forms

$$\Omega_i^j = dw_i^j + \sum_{k=1}^{n} w_k^j \wedge w_i^k. \tag{6}$$

Theorem 2.1 (Cartan structure equations)**.** The two Cartan structure equations are:

1. $dw^i = \sum_{k=1}^{n} w^k \wedge w_k^i$,
2. $\Omega_i^j = dw_i^j + \sum_{k=1}^{n} w_k^j \wedge w_i^k$.

2.3 Explicit Formula for $E = TM$

In fact, we can explicitly find a formula for the connection forms for $E = TM$. First, we define the dual bundle and dual frames:

Definition 2.4 (Dual bundle)**.** Given a vector bundle E over M. A dual vector bundle E^* is the bundle of linear maps $\mathrm{Hom}(E, \mathbb{R} \times M)$. In other words, for each $x \in M$, the fibre of E^* at x is the vector space of linear maps from the vector space $E(x)$ to the trivial fibre $\mathbb{R} \times \{x\}$.

Definition 2.5 (Dual frame)**.** Let $\{e_1(x), e_2(x), \ldots, e_n(x)\} \subset \Gamma(E)$ be an frame of a rank n vector bundle E. We define the dual frame to be an ordered collection of n smooth section of the dual bundle (or covectors) $\{w^1(x), w^2(x), \ldots, w^n(x)\} \in \Gamma(E^*)$ such that at each $x \in M$, $w^i(e_j) = \delta_j^i$ for all $i, j \in \{1, 2, \ldots, n\}$.

Remark 2.2. By using the rule in Remark 1.1 to define a connection on a dual space, we can deduce that for the dual frame $\{w^i\}$ we have

$$Dw^i = -\sum_{j=1}^{n} w^j \otimes w_j^i.$$

Remark 2.3. Using the first Cartan structure equation and a basis $\{e_i\}_{i=1}^n$ for $E = TM$, we would get the explicit formula for the connection forms:

$$w_i^k(e_j) = \frac{1}{2}(dw^i(e_j, e_k) + dw^j(e_i, e_k) - dw^k(e_i, e_j)), \tag{7}$$

where $\{w^i\}_{i=1}^n$ is the dual basis with respect to $\{e_i\}_{i=1}^n$ and d is the usual exterior derivative on the space of 1-forms.

2.4 Levi-Civita Connection and Riemannian Curvature Revisited

Now we have seen the general connection and curvature form, how do they relate to the Christoffel symbols and the Riemannian curvature in the first section?

Remark 2.4. Note that given a local coordinate $\{x^i\}$, we can construct a local orthonormal frame by Gram-Schmidt operation.

Now suppose that $\{\partial_i\}$ is the local orthonormal frame of the tangent bundle of an m dimensional Riemannian manifold (M, g). The Levi-Civita connection acts as:

$$\nabla_X \partial_i = \sum_{k=1}^{n} w_i^k(X)\partial_k$$

Thus, comparing with the formula for Christoffel symbols, we get that $w_i^k(e_j) = \Gamma_{ij}^k$. Furthermore, if we compare equation (7) with the formula for Christoffel symbols in (2), we note the similarity in both of these equations, with (7) being the more general one.

How about the curvature form? The curvature form in equation (6) coincides with the Riemann curvature tensor once we introduce a Riemannian metric g on M and use the Levi-Civita connection. In other words, the Riemannian curvature is derived from the curvature form by specifying the connection forms to be the Christoffel symbols. Indeed, Ω_j^i is a valued 2-form i.e. it is anti-symmetric in its arguments. It can be shown that

Theorem 2.2. We have:

$$\Omega_i^j(X, Y)\partial_j = R(X, Y)\partial_i.$$

Proof. Recall the definition for $R(X, Y)\partial_i$ and expand the expression:

$$\begin{aligned}
R(X, Y)\partial_i &= \nabla^2_{XY}\partial_i - \nabla^2_{YX}\partial_i \\
&= \nabla_X(\nabla_Y \partial_i) - \nabla_{\nabla_X Y}\partial_i - \nabla_Y(\nabla_X \partial_i) + \nabla_{\nabla_Y X}\partial_i \\
&= \nabla_X(w_i^j(Y)\partial_j) - w_i^j(\nabla_X Y)\partial_j - \nabla_Y(w_i^j(X)\partial_j) + w_i^j(\nabla_Y X)\partial_j \\
&= (\nabla_X w_i^j)(Y)\partial_j + w_i^j(Y)(\nabla_X \partial_j) - (\nabla_Y w_i^j)(X)\partial_j - w_i^j(X)(\nabla_Y \partial_j) \\
&= (\nabla_X w_i^j)(Y)\partial_j + w_i^j(Y)w_j^k(X)\partial_k - (\nabla_Y w_i^j)(X)\partial_j - w_i^j(X)w_j^k(Y)\partial_k \\
&= dw_i^j(X, Y)\partial_j + (w_j^k \wedge w_i^j)(X, Y)\partial_k \\
&= \Omega_i^j(X, Y)\partial_j.
\end{aligned}$$

by using the fact that $dw(X, Y) = (\nabla_X w)(Y) - (\nabla_Y w)(X)$ for any 1-form $w \in \Gamma(T^*M)$ and the definition for wedge product. □

Thus, finally, by taking the inner product of $\Omega_i^j(X, Y)\partial_j$ with ∂_j, since $\{\partial_i\}$ is chosen to be an orthonormal frame, we have $\Omega_i^j(X, Y) = g(R(X, Y)\partial_i, \partial_j)$. Note also that $\Omega_i^j = -\Omega_j^i$ by the antisymmetry in w_i^j and wedge products of 1-forms, confirming the anti-symmetry of the last two arguments in the Riemann curvature tensor.

3 An Application: Uniformisation Theorem via PDEs

Conformal class of metrics are metrics which are defined up to scale. Any local angles on the manifold M are the same for metrics within the same conformal class. However, other geometric properties, such as curvature and lengths, may differ.

Definition 3.1 (Conformal class of metrics)**.** A class of conformal metrics is a class of metrics which are equivalent under the relation: $h \sim g$ iff $h = ug$ for some positive smooth function $u : M \to \mathbb{R}$.

Suppose that (Σ, g) is a surface with Gauss curvature K. Consider all the metics $\tilde{g}$ in the conformal class of g i.e. $\tilde{g} = e^{2f} g$ for some smooth function $f : M \to \mathbb{R}$. The uniformisation theorem asserts:

Theorem 3.1 (Uniformisation theorem)**.** Every simply connected Riemannian surface (M, g) is conformally equivalent to a surface with constant Gaussian curvature of either $-1, 0$ or 1.

One way of proving this is using complex analysis. However, a more PDE oriented approach can be used. On the given manifold (M, g), let $\{e_1, e_2\}$ be an orthonormal frame for the tangent bundle with dual forms $\{w^1, w^2\}$. The first Cartan structure equation and the antisymmetry of w_j^i asserts that:

$$\begin{aligned} dw^1 &= w^2 \wedge w_2^1, \\ dw^2 &= w^1 \wedge w_1^2. \end{aligned}$$

Furthermore, since the only non-vanishing term in the Riemannian curvature tensor for (M, g) is $R_{1221} = K$, using the relation $\Omega_i^j(X, Y) = g(R(X, Y)\partial_i, \partial_j)$ and Cartan's second structural equation, we have:

$$dw_1^2 = -Kw^1 \wedge w^2. \tag{8}$$

Now we want to construct the same for the metric $\tilde{g}$. We note that an orthonormal dual frame for the metric $\tilde{g}$ is given by $\{\tilde{w}^1 = e^f w^1, \tilde{w}^2 = e^f w^2\}$. Thus, by Cartan's first structural equation, we have:

$$d\tilde{w}^1 = \tilde{w}^2 \wedge \tilde{w}_2^1. \tag{9}$$

On the other hand, by the definition of $\tilde{w}^1$, we have:

$$\begin{aligned} d\tilde{w}^1 &= df\, e^f \wedge w^1 + e^f dw^1 = (e_1(f)w^1 + e_2(f)w^2) \wedge \tilde{w}^1 + e^f(w^2 \wedge w_2^1) \\ &= e_2(f)\, \tilde{w}^2 \wedge w^1 + \tilde{w}^2 \wedge w_2^1 = \tilde{w}^2 \wedge (e_2(f)w^1 + w_2^1). \end{aligned} \tag{10}$$

Equating like terms in equations (9) and (10), we get $\tilde{w}_2^1 = e_2(f)w^1 + w_2^1$. Similarly, we get $\tilde{w}_1^2 = e_1(f)w^2 + w_1^2$. These look like good candidates for the expression of $\tilde{w}_1^2$ and $\tilde{w}_2^1$. However, recall that the connection forms must satisfy anti-symmetry i.e. $\tilde{w}_1^2 + \tilde{w}_2^1 = 0$. So, we amend the terms by adding a 0 term to the RHS of (10), and get:

$$\begin{aligned} \tilde{w}_1^2 &= (e_1(f)w^2 - e_2(f)w^1) + w_1^2, \\ \tilde{w}_1^2 &= (e_2(f)w^1 - e_1(f)w^2) + w_2^1. \end{aligned}$$

which satisfy the required antisymmetry. Now, recall equation (8) that on the Riemannian surface $(M, \tilde{g})$, we have

$$d\tilde{w}_1^2 = -\tilde{K}\tilde{w}^1 \wedge \tilde{w}^2.$$

But on the other hand, $d\tilde{w}_1^2 = (e_1^2(f) + e_2^2(f) - K)w^1 \wedge w^2$ and $\tilde{K}\tilde{w}^1 \wedge \tilde{w}^2 = \tilde{K}e^{2f}w^1 \wedge w^2$. Equating like terms, we get the PDE:

$$-\Delta_g f + K = \tilde{K}e^{2f}.$$

To prove the uniformisation theorem is is the same as proving the existence of solution of this PDE for $\tilde{K} = -1, 0, 1$.